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CHAPTER 11 CONTROL SYSTEMS DESIGN BY ROOT LOCUS

After completing this chapter, the students will be able to:

- Study adding zeros and/or poles on original root locus,
- Use root locus approach to design Lead and Lag compensators.

1. Introduction

Control systems are designed to perform specific tasks or requirements that generally called performance specifications. These specifications may be given in terms of transient response (such as the maximum overshoot and settling time in step response) and/or of steady-state requirements (such as steady-state error). The specifications of a control system must be given before the design process begins.

As the gain (K) is varied, both the transient and steady-state responses are also varied. Setting K at a particular value produces the transient response obtained by the poles at that point on the root locus. Thus, by changing K, we are limited to those responses that exist along the root locus.

For example, if certain transient response (defined by % overshoot, and settling time) is desired, and this transient response is represented by point B shown in Fig. 1. At the

specified % overshoot, we only can obtain the settling time represented by point A with the suitable gain adjustment. Thus, our goal is to speed up the response at A to that at point B. This cannot be accomplished by a simple gain adjustment, since point B does not lie on the root locus. This can be achieved by adding poles and zeros, so that the compensated system has a root locus that goes through the desired point B.
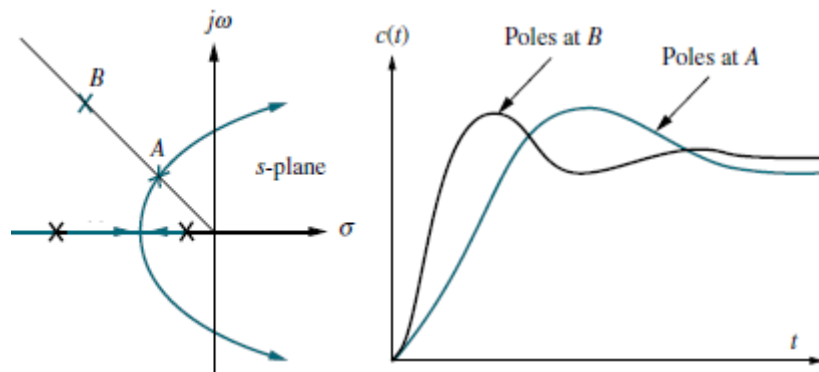

Fig. 1, Concept of design using root locus

The design by the root-locus method is based on reshaping the root locus of the system by adding poles and zeros to the system's open-loop transfer function and forcing the root locus to pass through desired closed-loop poles in the s plane. Adding pole or zero is done by a controller or compensator. The additional poles and zeros can be generated with a passive or an active network. Therefore, the compensator is a device inserted into the system for the purpose of satisfying the specifications. The compensator compensates for deficit performance of the original system. Commonly used compensators or controllers are:

- Lead compensator (PD controller)

- Lag compensator (PI controller)

- Lag-Lead compensator (PID controller)

2. Realization of Compensators

As we know, T.F. of an inverting operational amplifier whose configuration is given in Fig. 2. Is as follows:
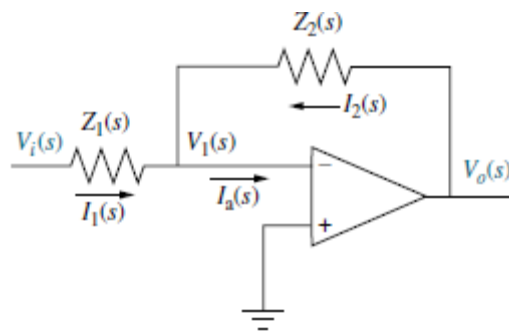
$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

Fig. 2, Operational amplifier configured for compensator realization

By the proper choice of $Z_1(s)$ and $Z_2(s)$, this circuit can be used as different kinds of controllers. Table below summarizes the realization of PI, PD, and PID controllers as well as lag, lead, and lag-lead compensators using operational amplifiers.

Function	Input impedance Z_1	Output impedance Z_1	T.F.
Gain	R_1	R_2	$-\dfrac{R_2}{R_1}$
I-Controller	R	C	$-\dfrac{\frac{1}{RC}}{s}$
D-Controller	C	R	$-RCs$
PI-Controller	R_1	R_2 C	$-\dfrac{R_2}{R_1}\dfrac{\left(s+\frac{1}{R_2C}\right)}{s}$
PD-Controller	C, R_1	R_2	$-R_2C\left(s+\dfrac{1}{R_1C}\right)$
PID-Controller	C_1, R_1	R_2 C_2	$-\left[\left(\dfrac{R_2}{R_1}+\dfrac{C_1}{C_2}\right)+R_2C_1s+\dfrac{\frac{1}{R_1C_2}}{s}\right]$
Lag Compensator	C_1, R_1	C_2, R_2	$-\dfrac{C_1}{C_2}\dfrac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_2C_2 > R_1C_1$
Lead Compensator	C_1, R_1	C_2, R_2	$-\dfrac{C_1}{C_2}\dfrac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_1C_1 > R_2C_2$

The T.F. of the compensator is $G_c(s)$ and may be connected in series as shown in Fig. 2-a) and is called series compensator. Also, it can be connected in feedback as shown in Fig. 2-b) and is called feedback compensator.
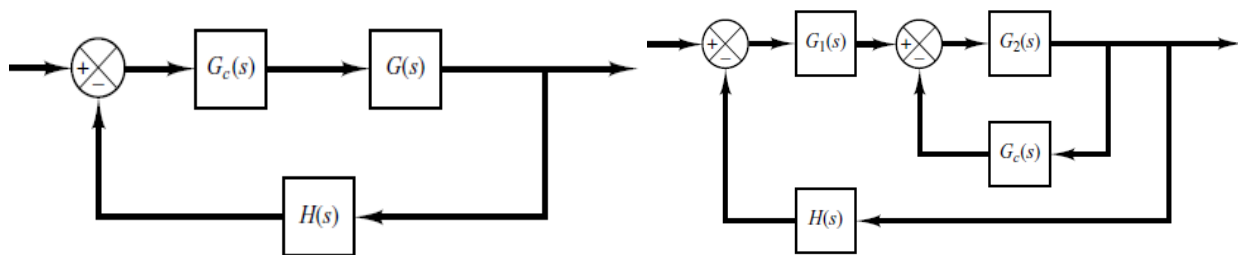

Fig. 2   a) series compensator   b) feedback compensator

3. Root-Locus Approach to Control System Design.

The root-locus method is a graphical method for determining the locations of all closed-loop poles based on the locations of the open-loop poles and zeros as the gain K is varied from zero to infinity. As an example, consider the system represented by

$$GH(S) = \frac{K}{(S+1)(S+2)}$$

As shown from the system root locus shown in Fig. 3, as the gain increases from K1 to K2 to K3 the damping ratio decreases affecting the transient performance but the steady-state error is improved.
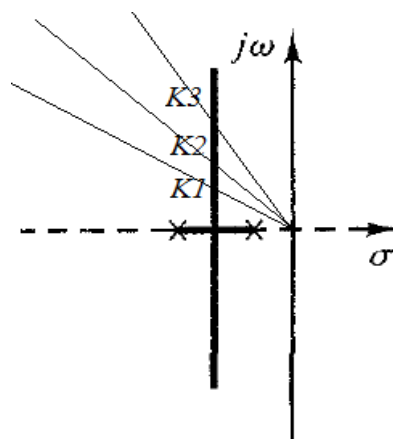

Fig. 2, root locus of 2nd order system

It is clear, when the system gain is adjusted to meet the transient response specification, steady-state error performance deteriorated, since both the transient response and the static error coefficient are related to the gain. The higher the gain, the smaller the

steady-state error, but the larger the percent overshoot. On the other hand, reducing gain to reduce overshoot increased the steady-state error. If we use compensators, we can meet transient and steady-state error specifications simultaneously.

2.1 Effect of Adding poles

The addition of a pole to the open-loop transfer function has the effect of pulling the root locus to the right, tending to lower the system's relative stability and to slow down the settling of the response. (Remember that the addition of integral part adds a pole at the origin, thus making the system less stable. On the other hand, it increases the system type and hence improve the steady state error.)

Figure 4 shows examples of root locus illustrating the effects of adding a pole or poles to a single-pole system and the addition of two poles to a single-pole system.

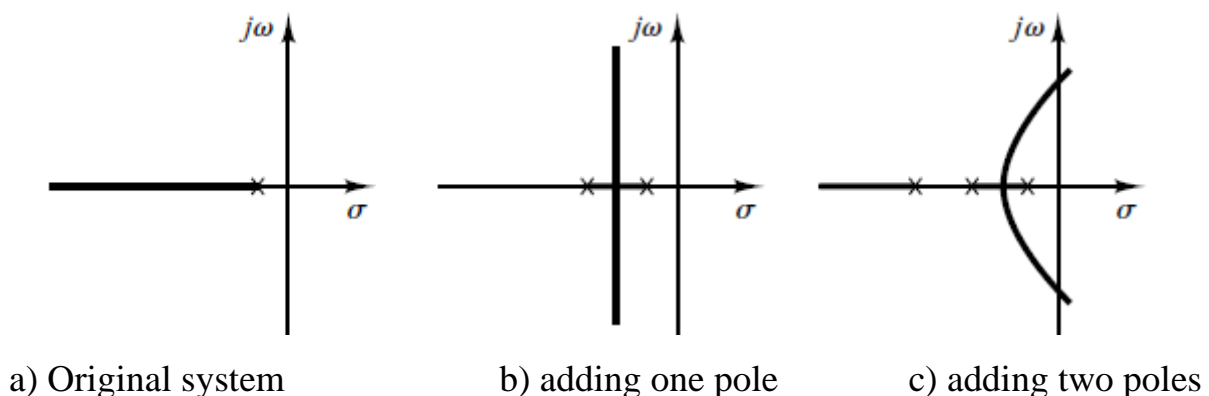a) Original system   b) adding one pole   c) adding two poles

Fig. 4, Effect of adding poles to control system

As the added pole becomes near to the origin, it has a great effect on reducing the system stability. Such kind of effect is reduced by moving the added pole to the left far from the origin.

2.2 Effects of Adding Zeros

The addition of a zero to the open-loop transfer function has the effect of pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response. Physically, the addition of a zero in the forward path means

the addition of derivative control to the system. The effect of such control is to enhance the transient response. Figure 5-a) shows the root locus for a system that is stable for small gain but unstable for large gain. Figures 5-b), c), and d) show root-locus plots for the system when a zero is added to the open-loop transfer function. Notice that when a zero is added to the system becomes stable for all values of gain.
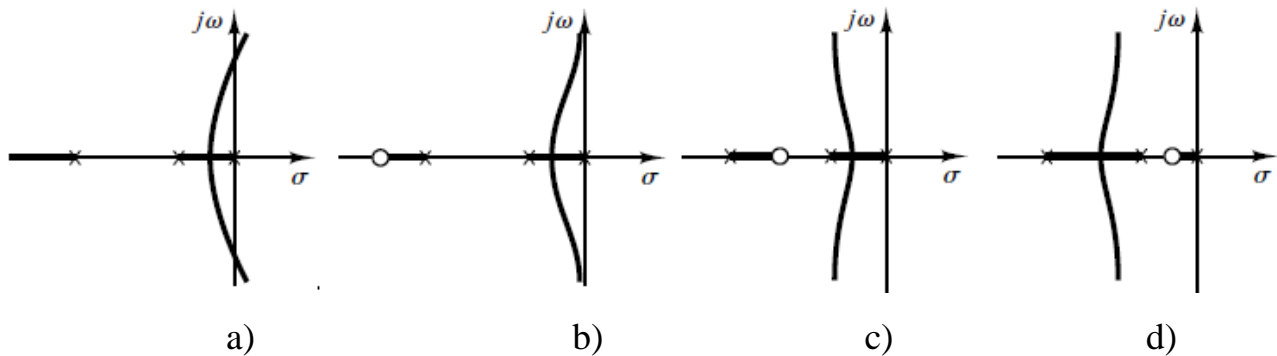

a)　　　　　　　b)　　　　　　　c)　　　　　　　d)

Fig. 5, Effect of adding zero

As the added zero becomes near to the origin, it has a great effect on enhancing the system transient response. Such kind of effect is reduced by moving the added zero to the left far from the origin.

3. **Phase-Lead Compensator**

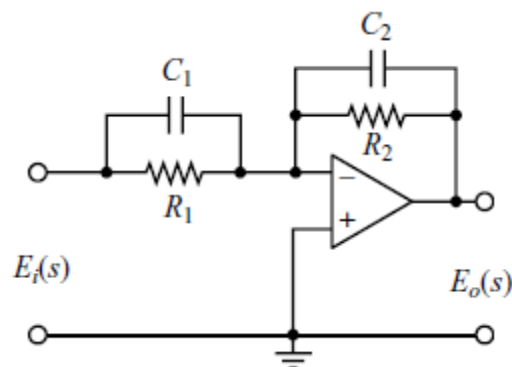The physical realization of the phase-lead compensator is shown in Fig. 6.

Fig. 6, Circuit realization of phase-lead compensator

Therefore, the lead compensator T.F. is given by:

$$G_c(S) = \frac{C_1}{C_2} \frac{S + \dfrac{1}{R_1 C_1}}{S + \dfrac{1}{R_2 C_2}} = \frac{S + Z_c}{S + P_c}$$

Where $R_1C_1 > R_2C_2$ also, Z_c and P_c are the zero and pole of the compensator which represented in the S-plane as shown in Fig. 7.
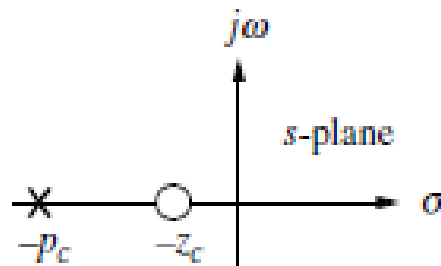

Fig. 7, Pole-zero map of lead compensator

It is clear from Fig. 7 that the $|Z_c| < |P_c|$. Therefore, the lead compensator named from its leading zero.

The T.F. of the lead compensator can be represented as:

$$G_c(S) = K_c \frac{(S + \frac{1}{T})}{(S + \frac{1}{\alpha T})}$$

Where α and T are constants. Also, α is +ve value less than unity. This means:

$$T = R_1 C_1 \quad and \; \alpha T = R_2 C_2$$

The procedure for designing a lead compensator for any control system is:

1. From the desired performance specifications, determine the desired location for the dominant closed-loop poles.

2. By drawing the root-locus of the uncompensated (original) system G(S), be sure that the gain adjustment alone can yield the desired closed-loop poles. If not, calculate the angle deficiency θ_c. This angle must be contributed by the lead compensator to reshape the new root locus is to pass through the desired locations obtained from step #1.

3. determine the value of α and T from the deficiency angle (θ_c).

To explain the above steps, consider the control system G(s) with two poles (P_1 & P_2) and one zero (Z_1) as given in Fig. 8.
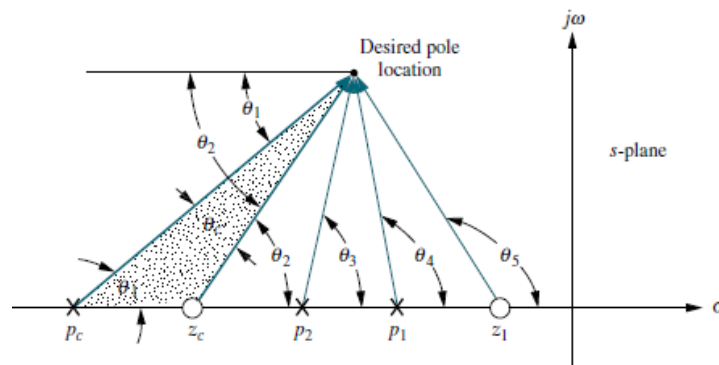
Fig. 8, Calculation of the lead compensator angle

Since the point of the desired pole location does not lie on the root locus of G(s). From the angle condition, we find that:

$$\theta_5 - (\theta_3 + \theta_4) < 180$$

Therefore, we need a phase lead compensator to add a positive angle θ_c which is:

$$\theta_c = \theta_2 - \theta_1$$

Therefore, the angle condition of the new root locus will be:

$$\theta_5 - (\theta_3 + \theta_4) + \theta_c = 180$$

Example 1

Design a lead compensator for the control system given in Fig. 9 so that the settling time (based on ±2%) is reduced to half value while maintaining 30% overshoot. Compare the time-domain characteristics of the system before and after the lead compensator.
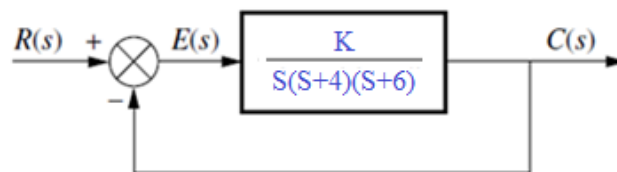

Fig. 9, Design example of lead compensator

First, draw root locus of the uncompensated system using the following Matlab code:

```
>> num=1;
>> den=[1 10 24 0];
>> sys=tf(num,den);
>> k=0:0.01:200;
>>rlocus(sys,k)
```
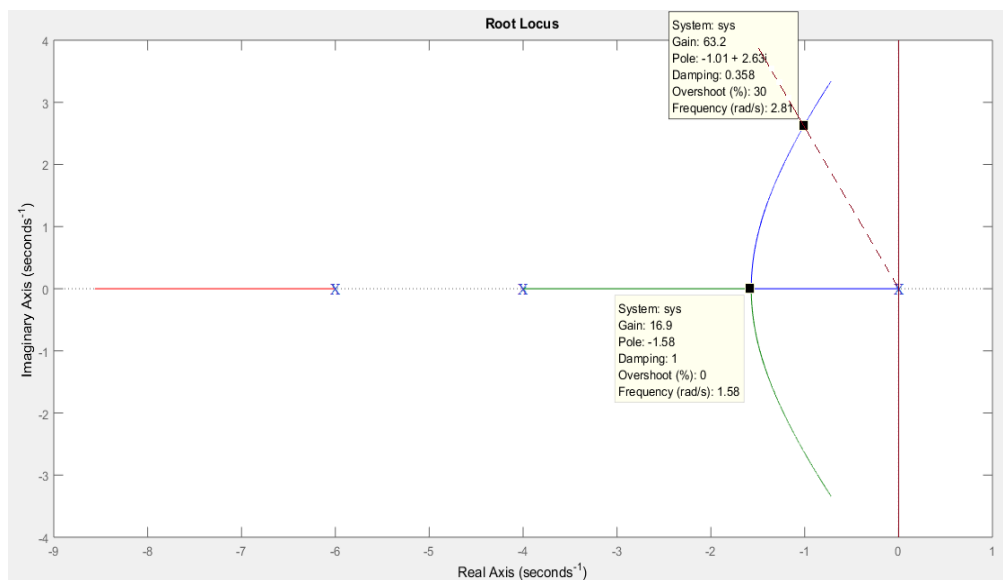The obtained root locus is shown in Fig. 10.

Fig. 10, Root locus of the uncompensated system

At the required maximum overshoot (30%), we find that $\zeta = 0.358$ and $\omega_n = 2.81$ rad/s.

The settling time (based on ±2%) is $4/(0.358 \times 2.81) = 3.976$ sec.

Now, it is required to reduce the settling time to half, i.e. $T_s = 1.988$ sec at the same ζ, so we can calculate the new value of $\omega_n = 5.62$ rad/s. The dominant pole is shifted from point A to point B as shown in Fig. 11.
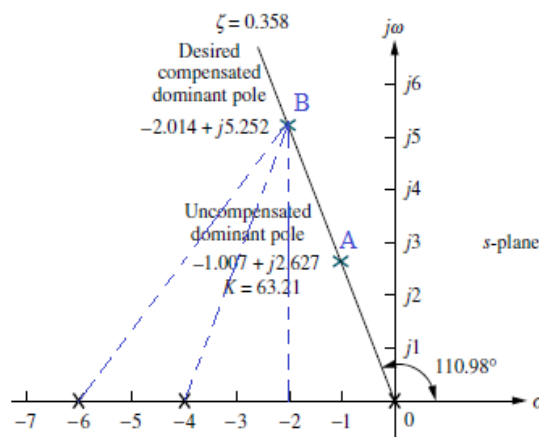

Fig. 11, Determination of the design point

The angle condition at point B is:

$$\theta = 0 - \left\{ 180 - tan^{-1}\frac{5.252}{2.014} + tan^{-1}\frac{5.252}{4 - 2.014} + tan^{-1}\frac{5.252}{6 - 2.014} \right\} = -233.07$$

Therefore, the compensator angle $(\theta_c) = -180 - (-233.07) = 53.07°$

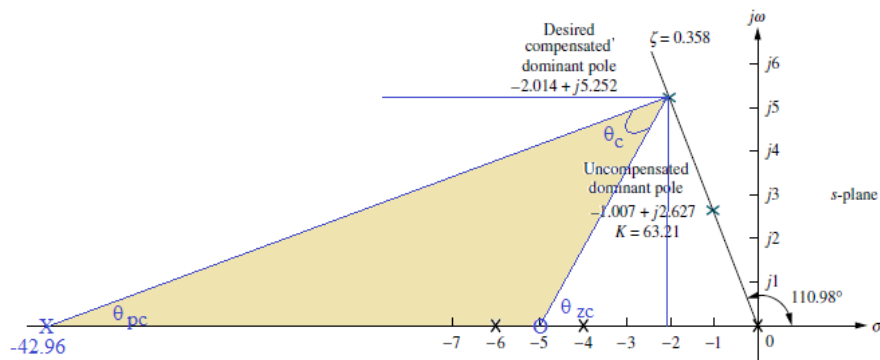Assume a compensator zero at –5 on the real axis as a possible solution (Fig. 12).

Fig. 12, Determination of pole and zero of lead compensator

$$\theta_{zc} = tan^{-1}\frac{5.252}{5 - 2.014} = 60.38$$

$$\theta_{pc} = \theta_{zc} - \theta_c = 60.38 - 53.07 = 7.31$$

$$\tan\theta_{pc} = 0.1283 = \frac{5.252}{P_c - 2.014} \rightarrow P_c = 42.96$$

Therefore, the location of the lead compensator pole (P_c) = 42.96

A comparison between the specifications of the uncompensated system with that of lead compensation is shown in table below.

system	Uncompensated	Lead Compensation
	$\dfrac{K}{s(s+4)(s+6)}$	$\dfrac{K(s+5)}{s(s+4)(s+6)(s+42.96)}$
Dominant poles	$-1.007 \pm j2.627$	$-2.014 \pm j5.252$
Other poles	-7.986	$-43.8, -5.134$
Zero	None	-5
K	63.21	1423
ζ	0.358	0.358
ω_n	2.813	5.625
$\%OS$	30	30
T_s	3.972	1.986
T_p	1.196	0.598
K_v	2.634	6.9
$e(\infty)$	0.380	0.145

Example 2:
Consider the unity feedback control system:

$$G(S) = \frac{10}{S(S + 1)}$$

It is desired to make the system damping ratio 0.5 with undamped natural frequency 3.0 rad/s. Design the suitable lead compensator and draw the root locus of the compensated system.

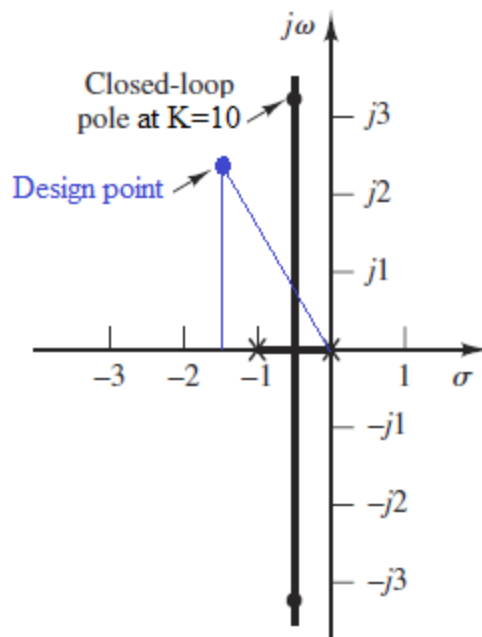1) Draw the root locus of the original system as shown in Fig. 13.

Fig. 13, Root locus of the original system indicating the design point

At K=10, the closed loop poles are located at $-0.5 \pm J3.1225$.

At $\xi = 0.5$ & $\omega_n = 3$ it is found that the closed loop poles are located at $-1.5 \pm J2.5981$,

$$s^2 + 2\zeta \omega_n s + \omega_n^2 = s^2 + 3s + 9$$
$$= (s + 1.5 + j2.5981)(s + 1.5 - j2.5981)$$
$$s = -1.5 \pm j2.5981$$

2. Compute the angle at the design point shown in Fig. 13.

$$\theta = 0 - \left\{180 - tan^{-1} \frac{2.5981}{1.5} + 180 - tan^{-1} \frac{2.5981}{1.5 - 1}\right\} = -220.894$$

But for the root locus the angle must be ± 180, therefore, if we need to force the root locus to go through the desired closed-loop pole, the lead compensator must contribute

$$\theta_c = -180 - (-220.894) = 40.894$$

at the design point.

3- Using the bisector method, explained in Fig. 14, determines the location of the compensator pole and zero.

$$G_c(S) = K_c \frac{(S + \frac{1}{T})}{(S + \frac{1}{\alpha T})}, \quad 0 < \alpha < 1$$

At the design point P connect OP and draw the horizontal line AP. Then measure the angle OPA. Draw the bisector PB so that it divides the angle OPA equally. From PB measure half the lead angle θ_c before and after the bisector to get the line PC and PD, respectively.
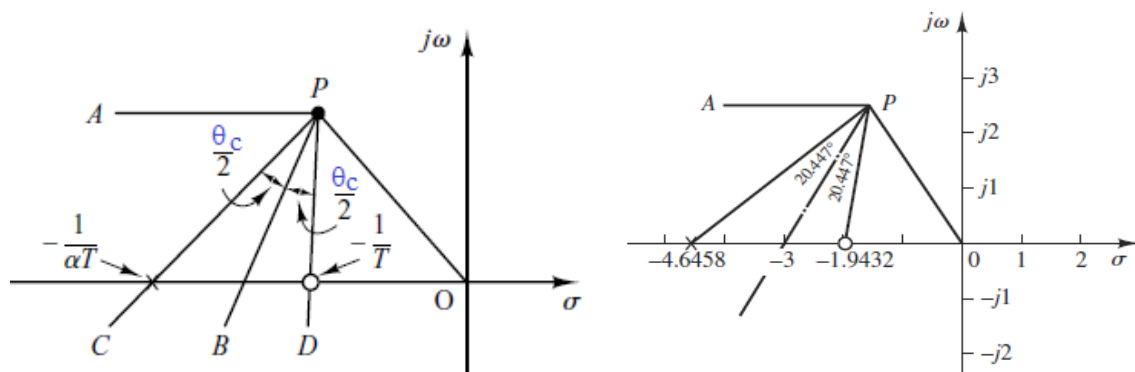

Fig. 14, Bisector method to determine the lead pole and zero

So, we obtain that

Zero at S=−1.9432

Pole at S=−4.6458

And α = 1.9432 ÷ 4.6458 = 0.418

Now the compensated system become

$$G_c(s)G(s) = \frac{K_c(S + 1.9432)}{(S + 4.6458)} \times \frac{10}{S(S + 1)} = \frac{10K_c(S + 1.9432)}{S(S + 1)(S + 4.6458)}$$

The value of K_c is determined from the magnitude condition:

$$\left| K_c \frac{s + 1.9432}{s + 4.6458} \frac{10}{s(s + 1)} \right|_{s=-1.5+j2.5981.} = 1$$

$$K_c = \left| \frac{(s + 4.6458)s(s + 1)}{10(s + 1.9432)} \right|_{s=-1.5+j2.5981.} = 1.2287$$

The compensated system becomes

$$G_c(s)G(s) = \frac{12.287(S + 1.9432)}{S(S + 1)(S + 4.6458)}$$

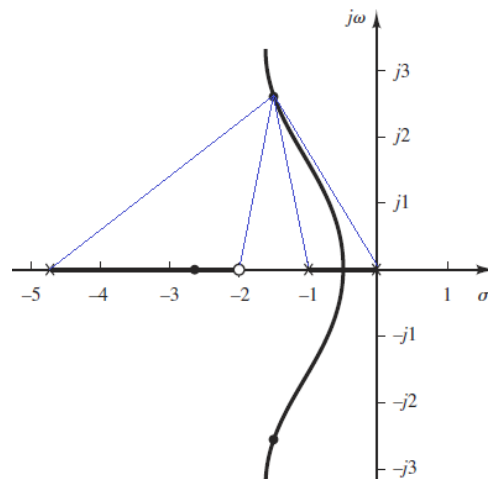The root locus of the compensated system is shown in Fig. 15.

Fig. 15, Root locus of the compensated system

4. Lag Compensator

The lag compensator T.F. is given by

$$G_c(S) = K_c \frac{(S + \frac{1}{T})}{(S + \frac{1}{\beta T})}$$

Where α and T are constants. Also β is greater than unity.

We select the lag compensator when the system exhibits satisfactory transient-response characteristics but unsatisfactory steady-state characteristics. Compensation in this case essentially consists of increasing the open-loop gain without appreciably changing the transient-response characteristics. This means that the root locus in the neighborhood of the dominant closed-loop poles should not be changed appreciably, but the open-loop gain should be increased as much as needed.

This can be accomplished if a lag compensator is put in cascade with the given feed forward transfer function. To avoid an appreciable change in the root locus, the angle contribution of the lag network should be limited to a small amount, say 5°. To assure this, we place the pole and zero of the lag network relatively close together and near the origin of the *s* plane. Then the closed-loop poles of the compensated system will be shifted only slightly from their original locations. Hence, the transient-response characteristics will be changed only slightly.

$$-5° < \left| \frac{s_1 + \frac{1}{T}}{s_1 + \frac{1}{\beta T}} \right| < 0°$$

Example 2

Consider the unity feedback control system:

$$G(S) = \frac{K}{S(S + 1)(S + 2)}$$

At a damping ratio of 0.491, it is required to make the velocity error coefficient = 5,

Design the suitable lag compensator & draw the root locus of the compensated system.

From the root locus of the uncompensated system, the closed loop poles at $\xi = 0.491$ are located at S = -0.3307 ± J 0.5864 as shown in Fig. 8.

From the magnitude condition, K = 1.06

The static velocity error coefficient K_v = 1.06 / 2 = 0.53

The required velocity error coefficient K_v is 5, this mean,

$$K_v = 5 = \lim S \frac{1.06 \left(S + \frac{1}{T}\right)}{S(S + 1)(S + 2)\left(S + \frac{1}{\beta T}\right)} = \frac{1.06 \left(\frac{1}{T}\right)}{2\left(\frac{1}{\beta T}\right)} = 0.53\, \beta$$
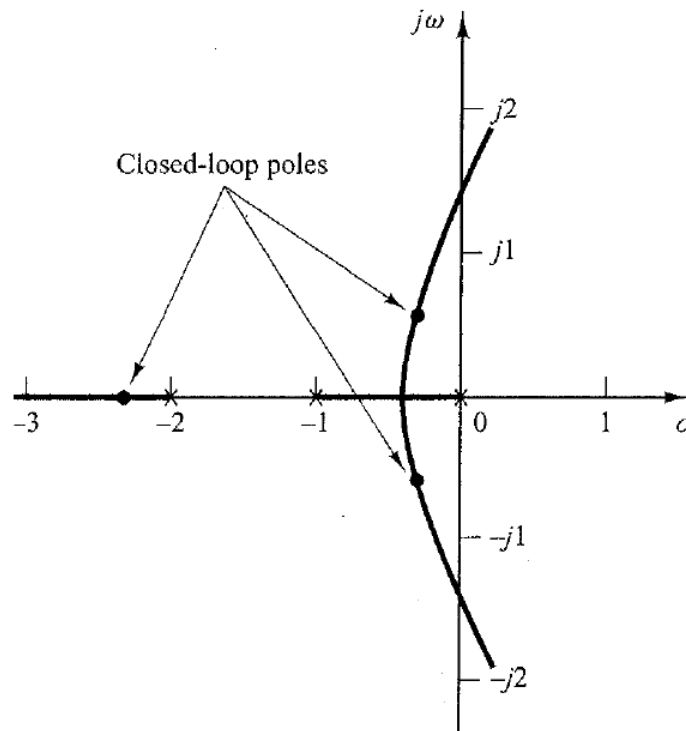
β = 9.434

Fig. 8, root locus of uncompensated system

Let T = 2β = 2 × 9.434 = 18.868

Therefore, the lag zero is located at 1/T = 0.053

And the lag pole is located at 1/βT = 0.0056

The compensated system is

$$G(S) = \frac{1.06\, K_c (S + 0.053)}{S(S + 1)(S + 2)(S + 0.0056)}$$
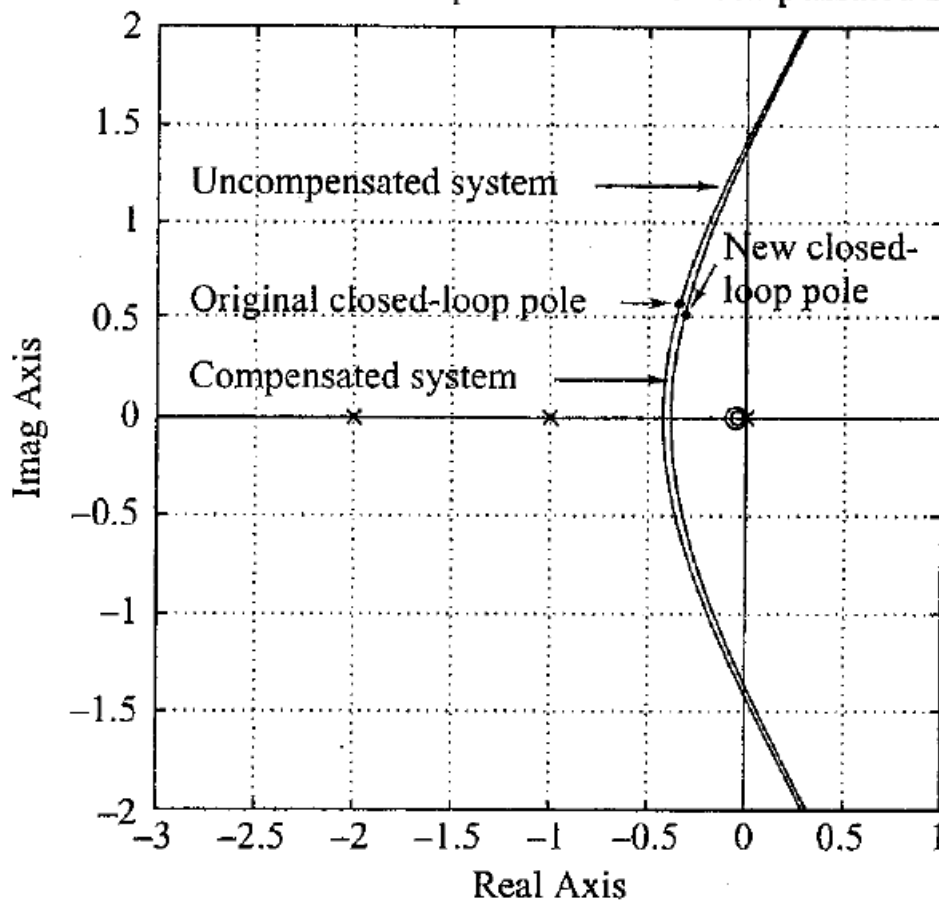
The root locus of the compensated system is shown in Fig. 9.

At the same value of the damping ratio of 0.491, the gain is calculated as

1.06 K$_c$ = 1.0235

K$_c$ = 0.966

Root-Locus Plots of Compensated and Uncompensated Systems

Sheet (9) Design by Root Locus

Problem (1)

Determine the values of K, T_1 and T_2 of the system shown in Fig. 1 so that the dominant closed-loop poles have a damping ratio $\xi = 0.5$ and the undamped natural frequency = 3 rad/sec.
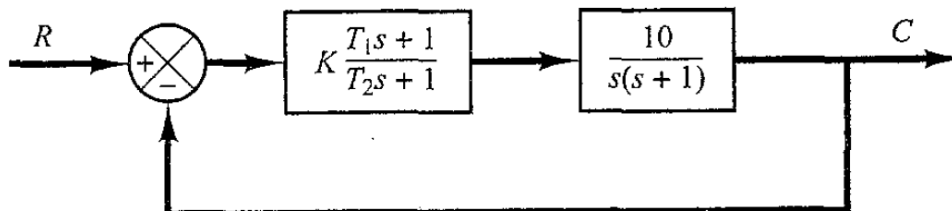

Fig. 1, Control system for problem 1

Problem (2)

Consider the control system shown in Fig. 2, design a lead compensator such that the dominant closed-loop poles are located at $S = -2\pm J2\sqrt{3}$
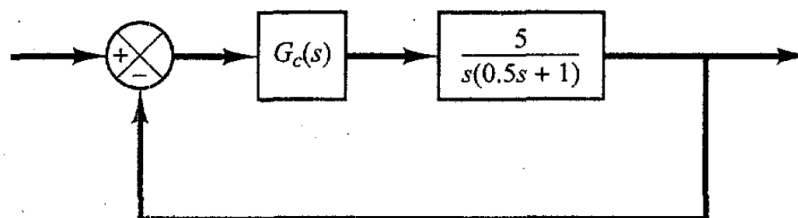

Fig. 2, Control system for problem 2

Problem (3)

Consider the control system shown in Fig. 3, find the compensator parameters so that the dominant closed-loop poles are located at $-1\pm J1$
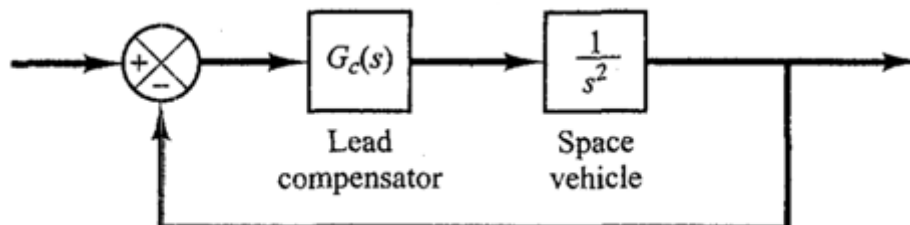

Fig. 3, Control system for problem 3

Problem (4)

For the control system shown in Fig. 4, design a compensator such that the static velocity error coefficient K_v is 20 sec^{-1} without appreciably changing the original location ($S = -2 \pm J\,2\sqrt{3}$) of a pair of the complex conjugate closed-loop poles.
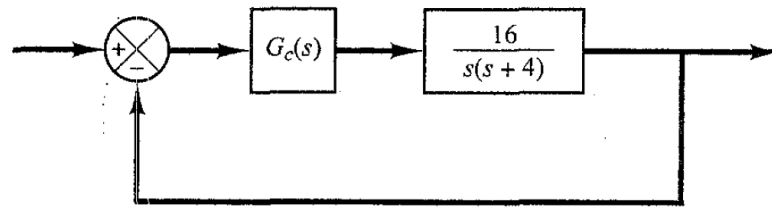
Fig. 4, Control system for problem 4

Problem (5)

Consider the control system shown in Fig. 5. The plant is critically stable at the defined value of gain. Design a suitable compensator such that the unit-step response will exhibit maximum overshoot of less than 40% and a settling time of 5 sec or less.
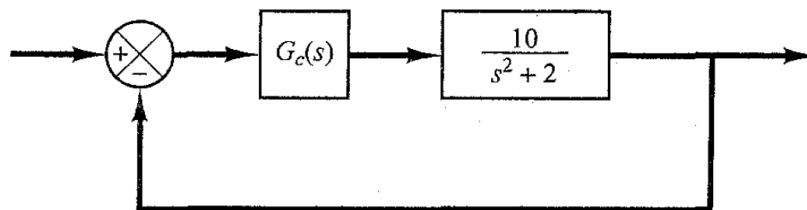

Fig. 5, Control system for problem 5

Problem (6)

Consider the control system shown in Fig. 6. Determine the value of K and T of the controller $G_c(S)$ such that the closed-loop poles are located at $S = -2 \pm J2$
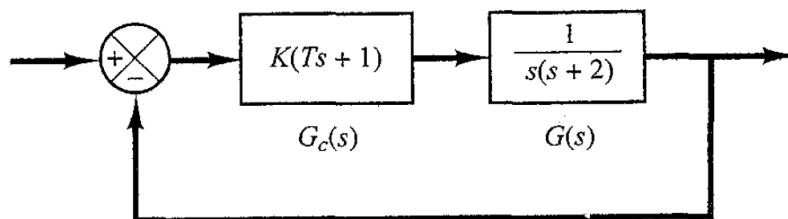

Fig. 6, Control system for problem 6

References:

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